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BOUNDED PHASE-COORDINATE CONTROL

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INTRODUCTION

This is the third quarterly progress report prepared for the National Aeronautics and Space Administration under Contract No. NASw-986. The body of this report is qualitative in nature. Preliminary versions of detailed technical reports are given as appendices. The final report will contain a complete and detailed description and summary of the work carried on under the contract.

The objective of the research being carried out is the development of methods by means of which bounded phase-coordinate controllers for large flexible launch boosters can be realized. The areas for investigation are the determination of the zero cost sets, the determination of extremal controls, and computational procedures for finding optimal controls.

SUMMARY OF ACCOMPLISHMENTS

The main effort during the third quarter was applied to the implementation of Neustadt's algorithm on an analog computer for the solution of the bounded phase-coordinate problem. It was found that the algorithm could be set up on the analog for the time-optimal problem. This is reported in Appendix A. However, because the constrained subarcs are singular, the method failed when the phase constraints are added. The details are supplied in Appendix B.

The state of work on the general problem is outlined in Appendix C with suggestions for work for the next quarter. The analysis of Appendix A from the Second Quarterly Report has been revised and is submitted as Appendix D. This work was presented in a seminar at the National Aeronautics and Space Administration Marshall Space Flight Center, February 2-4. It will appear in "Progress Report No. 7 on Studies in the Fields of Space Flight and Guidance Theory" issued by the Astrodynamics and Guidance Theory Division at Marshall.

APPENDIX A

MECHANIZATION OF NEUSTADT'S ALGORITHM FOR TIME-OPTIMAL CONTROL ON AN ANALOG COMPUTER

by

D. D. Fairchild

MECHANIZATION OF NEUSTADT'S ALGORITHM
FOR TIME-OPTIMAL CONTROL ON AN ANALOG COMPUTER

by

D. D. Fairchild

The technique which is described allows "on line" simulation of the time-optimal regulator by adapting Neustadt's algorithm to analog computation. The procedure is an out-growth of attempts to realize bounded-phase coordinate controllers.

THE PROBLEM

Given the equations of the system in the form of an n^{th} order vector differential equation

$$\dot{x} = A(t)x + B(t)u(t) \tag{1}$$

we must find a control vector $u^*(t)$, of r components, which steers the system from an initial state x_o at $t = 0$, to a final position in which all components of x are zero, with the finite time of transit T being a minimum. The additional proviso that

$$|u_j(t)| \leq 1; \; j=1,2,\ldots m$$

must be added to set the problem.

NEUSTADT'S STRATEGY

The variation of parameters formula gives

$$x(t) = X(t)x_o + X(t)\int_0^t X^{-1}(s)B(s)u(s)ds \tag{2}$$

as a representation of the solution of equation (1). The matrix X(t) is the matrix solution of the homogeneous part of (1) which becomes the identity matrix for t=0. If $x(t_o) = 0$, multiplying equation (1) by $X^{-1}(t_o)$ yields the formula

$$-x_o = +\int_0^{t_o} X^{-1}(s)B(s)u(s)ds. \tag{3}$$

This may be interpreted as producing the set of initial conditions x_o from which the origin can be reached in t_o seconds by application of control function u. Taking the dot product of (3) with a vector η, yet to be determined, we have

$$-\eta \cdot x_o = \int_0^{t_o} \eta \cdot X^{-1}(s)B(s)u(s)ds. \tag{4}$$

By selecting

$$u(s) = \mathrm{Sgn}\left[\eta \cdot X^{-1}(s)B(s)\right] \tag{5}$$

the expression given in (4) is made maximum for all η. The time optimal regulator for normal systems is assured (Ref 3) for a particular value $\eta = \eta_o$. To obtain Neustadt's relationship, it is necessary to define the following:

$$Z(t,\eta) = \int_0^{t} X^{-1}(s)B(s)\ \mathrm{sgn}\left[\eta \cdot X^{-1}(s)B(s)\right]ds \tag{6}$$

Making use of (6), equation (4) may be written

$$-\eta \cdot x_o = \eta \cdot Z(t,\eta), \text{ when } t = t_o. \tag{7}$$

Expression (7) may be written as

$$0 = \eta \cdot \left[Z(t_o,\eta) + x_o\right]. \tag{8}$$

Equation (8) is satisfied by the η corresponding to the time optimal regulator for the initial condition x_o. Neustadt considered the function

$f(t,\eta;x_o) = \eta\cdot[Z(t,\eta) + x_o]$, and proved the following properties (9)

a) $f(t,\eta;x_o)$ is continuous in t and η, and

b) $f(t,\eta;x_o)$ is strictly increasing with t for a fixed η.

Further insight to the significance of (9) can be gained by graphical arguments. Using (3) it is possible to construct a graph of the set of all initial conditions from which the origin can be reached in t seconds. Such a graph is shown in Fig. 1 for $t_1 < t_2 < t_3 < t_o$. Selecting η_a arbitrarily, the corresponding $Z_a(t,\eta)$ is constructed in Fig. 1. Examination of (9) reveals that $f(t,\eta;x_o)$ may be reduced to zero by either of two means:*

a) causing the vectors η and $[Z(t,\eta)+x_o]$ to form a right angle, or,

b) reduction of the vector $[Z(t,\eta)+x_o]$ to zero.

Returning to Fig. 1., the vector $[Z_a(t,\eta) + x_o]$ is constructed for the particular η_a shown. It is apparent that the angle between η_a and $[Z_a(t,\eta) + x_o]$ (i.e., angle θ) will be 90^o for some time $t_1 < t_a < t_3$. Thus the first of two conditions necessary for (9) to be zero has been illustrated. It can be further observed that

*The discussion which follows assumes that $f(t,\eta;x_o)$ is initially negative, which is equivalent to saying that the angle between x_o and η is greater than 90^o.

the projection of $[Z_a(t,\eta)+ x_o]$ on η_a will be negative for $t < t_a$, and will become positive for $t > t_a$ for a fixed x_o.

The second condition (i.e., reducing $[Z(t,\eta)+x_o]$ to zero) is only possible when $t = t_o$. In addition, $Z(t_o,\eta;x_o)$ must be coincident with negative x_o, which fixes the corresponding η_o.

The vector $[Z(t,\eta)+x_o]$ will be nonzero for all $t \neq t_o$, and the projection of η will be outward or positive for all $t > t_o$. Therefore $t = t_o$ is the upper bound of the zero crossings of $f(t;\eta,x_o)$ considered as a function of t. In other words, $t = t_o$ is the upper bound of ω where $\omega = \{T \mid f(T;\eta;x_o) = 0\}$, and further $t_o \in \omega$.

Plots of $f(t,\eta;x_o)$ versus time for several values of η, as obtained from the preceeding graphical argument, are given in Fig. 2.

IMPLEMENTATION

Utilizing a circuit which permits maximizing $T\epsilon\omega$, the optimal controller corresponding to a given initial condition can be obtained.

The Bang-Bang or Coulomb Friction Circuit driven by $f(t,\eta;x_o)$ can provide an output of the form

$$\begin{aligned} V(t,\eta;x_o) = e_{o_{(BB)}} &= k, \qquad \text{for } f(t,\eta;x_o) < 0, \\ e_{o_{(BB)}} &= 0, \qquad \text{for } f(t,\eta;x_o) > 0. \end{aligned} \tag{10}$$

By driving an integrator with the output of the Bang-Bang circuit, the output voltage of the integrator will be of the form

$$U(t,\eta;x_o) = e_{o(int)} = kt, \quad \text{for } t \leq T$$
$$= 0 + kT, \quad \text{for } t > T; \tag{11}$$

where T is a particular time such that $f(T,\eta;x_o) = 0$. For a given initial condition x_o and a trial value η, (11) is only a function of time. By changing η and repeating the solution, it is possible to obtain a set of curves $U(t,\eta;x_o)$ which are a function of the time at which $f(t,\eta;x_o)$ is zero. A graph of the set of curves $U(T,\eta;x_o)$ is shown in Fig. 3. To generate $U(T,\eta;x_o)$ with the computer, the form of (9) was modified to

$$\bar{f}(t,\eta;x_o) = \eta(t) \cdot x(t), \tag{12}$$

which is shown equivalent to Neustadt's expression by the following argument.

If $\eta(t) = [X^T(t)]^{-1}\eta_o$, then

$$\eta(t) \cdot x(t) = [X^T(t)]^{-1}\eta_o \cdot [X(t)x_o + X(t)\int_0^t X^{-1}(s)B(s)u(s)ds] \tag{13}$$

LaSalle (Ref 3) proves the optimal steering function to be of the form

$$u(s) = \text{sgn}[\eta(t) \cdot X(t)X^{-1}(s)B(s)], \qquad 0 \leq s \leq t,$$

which is equivalent to

$$u(s) = \mathrm{sgn}[\eta_o \cdot X^{-1}(s)B(s)]. \tag{14}$$

Substituting (14) into (13) we have

$$\eta(t) \cdot x(t) = \eta_o \cdot [x_o + \int_0^t X^{-1}(s)B(s)\mathrm{sgn}[\eta_o \cdot X^{-1}(s)B(s)]ds, \tag{15}$$

but

$$\int_0^t X^{-1}(s)B(s)\ \mathrm{sgn}[\eta_o \cdot X^{-1}(s)B(s)]ds = Z(t,\eta;x_o).$$

Therefore (15) can be expressed as

$$\eta(t) \cdot x(t) = \eta_o \cdot [x_o + Z(t,\eta;x_o)], \text{ which} \tag{16}$$

demonstrates that $\bar{f}(t,\eta;x_o) = f(t,\eta;x_o)$.

Advantage was taken of the repetitive solution capabilities of the REAC-C400 analog computer. In this mode the circuitry functions such that during one half of a square wave cycle the integrator capacitor terminals are "shorted" to discharge the capacitor, and during the other half cycle the integrator is placed in "Operate". Thus the computer re-solves the program at a frequency determined by an external square wave generator. The repetitive solutions are then displayed on an oscilloscope, such as the Electromec large-screen oscilloscope. By varying the initial values of $\eta_{j(o)}$; $j=0,1,2,\ldots m$, a continuous display similar to that of Fig. 3 is available. The effect of each new setting of the elements of η is immediately apparent and maximization of $\omega(T)$ is facilitated with a minimum of interpretation required.

APPLICATION

The intended application of Neustadt's algorithm was in conjunction with the "soft bounded" phase coordinate problem, more properly termed "the approximate linear time optimal control process with bounded phase coordinates." The problem statement given in (1) is still applicable with the additional requirement that $x_j(t)$, $j = 1,2,\ldots m$ remain within a given constraint during its response.

The equations defining the specific case studied (i.e., an augmented harmonic oscillator) are

$$\dot{x}_o = F(x_2)$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + \operatorname{sgn} \eta_2(t)$$

$$\dot{\eta}_o = 0$$

$$\dot{\eta}_1 = \eta_2$$

$$\dot{\eta}_2 = -\eta_1 - \eta_o \frac{\partial F(x_2)}{\partial x_2} \quad ,$$

where
$$F(x_2) = \tfrac{1}{2}(x_2 - \tfrac{1}{2})^2, \qquad \text{if } x_2 \geq \tfrac{1}{2}$$
$$= 0 \,, \qquad \text{if } |x_2| < \tfrac{1}{2}$$
$$= \tfrac{1}{2}(x_2 + \tfrac{1}{2})^2 \,, \qquad \text{if } x_2 \leq \tfrac{1}{2}.$$

The parameters subscripted with a zero result from augmenting the system to permit enforcing the "soft boundary" (Ref 2). The augmented system was believed to be normal (Ref 2), however this assumption later proved to be invalid. The systems non-normality was initially indicated by the potentiometer settings required to maximize the algorithm (i.e., η_o had to be zero which corresponds to the unbounded case). The level of confidence in these first indications was improved when attempts to determine the required initial values of the elements of η using grid networks were also unsuccessful. Increments of $\frac{1}{100}$ over a range from 1 to 10 were used in the grid networks search. The phase coordinates of the state vectors were plotted for each trial in the grid network. No trial combination within the grid network resulted in switching or tracking along the boundary. Further refinement of the grids was not considered worthwhile. These simulating results warranted further analytic studies which proved that the coordinate $\eta(t)$ vanished for a finite time while tracking along the boundary, leaving the controller undefined and non-extremal. Having established that the controller is undefined over a portion of the switching boundary invalidates the mechanization scheme being used. It is interesting to note that the technique "recognized" this condition by only providing data for the unbounded case. Recall that normal systems are required to have no component of $[\eta \cdot X^{-1}(t)B(t)]$ vanish on any interval, with $\eta \neq 0$ (Ref 3).

A detailed treatment of the soft bounded problem will be left to those more qualified, since the applicability of the technique under discussion is void for non-normal systems (Ref 1).

In implementing the system of (10) it was observed that the solution obtained with $\eta_o = 0$ was the correct solution to the unbounded problem. This result was anticipated since the systems equations reduce to the unbounded case in that configuration. Several initial conditions of the state variables were investigated, and the required initial conditions of the adjoint vectors were obtained. It should be observed at this point that the selection of initial conditions of the state vectors was conditioned by the particular system of equations, and a desire to compare the results obtained with those obtained by Russell (Ref 4) for the same system. The conditions investigated were in the range

$$1 \leq x_1(0) \leq 2, \quad \text{and}$$

$$x_2(0) = 0.$$

It is hoped that additional results with $x_1(0) > 2$ and $x_2(0) \neq 0$ can be investigated subsequent to the writing of this memo.

The analog computer program used to simulate the set of equations of (10) and implement Neustadt's algorithm is shown in Fig. 4. Two examples of the phase plane plots are given in Fig. 5. For the examples of Fig. 5, the time variations of individual parameters are shown in Fig. 6.

CONCLUSIONS

The conclusions reached in this study are as follows:

1) the technique is not applicable to the bounded phase coordinate problem as that problem is presently stated,

2) "on line" applications do exist for the unbounded phase coordinate problem, and

3) where applications exist, complete mechanization of the search procedure (i.e., removing the operator from the loop) should be considered. Open loop versus closed loop automation also has interesting implications.

The results demonstrated that on line time optimal control could be obtained for possibly higher than second order systems when the plant dynamics are slow, as in chemical reactor control problems where one or two minutes can be spent in obtaining a feasible solution. Also, in such applications a digital computer (of perhaps the Honeywell 200 class) could be programmed to seek the minimum, and thereby completely mechanize the search procedure.

Further, the method as it stands could be used to obtain feasible solutions as needed in training the feedback controller of the logic net mechanization [Ref 5], or other such applications.

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APPENDIX B

ANALOG COMPUTATION OF TIME OPTIMAL CONTROL FOR APPROXIMATE BOUNDED PHASE-COORDINATE SYSTEM

by

J. Y. S. Luh

ANALOG COMPUTATION OF TIME OPTIMAL CONTROL FOR APPROXIMATE BOUNDED PHASE-COORDINATE SYSTEM

by

J. Y. S. Luh

ABSTRACT

The synthesis method for approximate bounded phase-coordinate time optimal control [Reference 1] was studied on an analog computer. The results indicated that the method cannot be directly implemented. Recommendations are made for further study in connection with the singular arc.

INTRODUCTION

As shown in Reference 1, the subject problem can be stated as follows:

Consider a linear control process described by the system of differential equations

$$\dot{x} = A(t)x + B(t)u(t),$$

where the coefficient matrices A(t) and B(t) are composed of known continuous functions on the time interval $[t_o, t_1]$. Find an allowable controller u(t) which steers $x_u(t)$ from x_o at t_o to a prescribed compact target set $\mathfrak{T}$, with $x_u^o(t_1) \leq \beta$ and $t_1 - t_o$ a minimum, where

$$x_u^o(t_1) = \int_{t_o}^{t_1} F(x_u(t))dt,$$

F(x) = convex continuous differentiable function such that

$$F(x) \begin{cases} = 0, & \text{if } x \text{ remains within a given constraint set } \Lambda; \\ \neq 0, & \text{otherwise.} \end{cases}$$

To solve the problem, the proposed method [Ref. 1] considers an augmented 2n+2 system of equations

$$
S)\begin{cases}
\dot{x}^{o} = F(x) \\
\dot{x} = A(t)x + B(t)\,\mathrm{sgn}\,\{\eta\, B(t)\} \\
\dot{\eta}^{o} = 0 \\
\dot{\eta} = -\eta\, A(t) - \eta^{o}\,\dfrac{\partial F'(x)}{\partial x},
\end{cases}
$$

with $x^{o}(t_o) = 0$, $x(t_o) = x_o$, and $\eta^{o}(t_o) < 0$. Let

$$\hat{x} = (x^{o}, x),$$

$$\hat{\eta} = (\eta^{o}, \eta),$$

and $\hat{f} = \hat{\eta}[\hat{x} - \alpha]$,

where

$$\alpha = \begin{bmatrix} \beta \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then the system S) could be solved on an analog computer by means of modified Neustadt's algorithm [Reference 2].

EXPERIMENTAL RESULTS

Two examples were studied on the analog computer:

(a) Pure inertia system $\ddot{x}_1 = u$ with $\Lambda = |\dot{x}_1| \leq 0.5$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \dot{x}_1 \end{bmatrix},$$

$$F(x) = \begin{cases} 0.5(x_2 - 0.5)^2 & \text{for } x_2 \geq 0.5 \\ 0 & |x_2| \leq 0.5 \\ 0.5(x_2 + 0.5)^2 & x_2 \leq -.05 \end{cases}$$

$$\text{Target} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Various initial conditions $x(0)$ and different values of β were considered on the computation. A set of typical phase-coordinate trajectories with $x(0) = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}$, $\beta = 0.5$, and various trials of $\hat{\eta}(0)$ is shown in Figure 1. Note that the trajectory which supposedly corresponds to a time optimal* solution does not pass through the origin, and the trajectory which ends at the origin does not correspond to a time optimal solution. Moreover, there is no indication showing the effectiveness of the phase-coordinate constraint.

(b) Harmonic oscillator system $\ddot{x}_1 + x_1 = u$ with $\Lambda = |\dot{x}_1| \leq 0.5$

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*A time optimal solution [Ref. 2] is determined as follows: Select $\eta^\circ(0)$ and $\eta(0)$ such that $\hat{f}(t)|_{t=0} < 0$. Integrate the system until $\hat{f}(T) = 0$. Thus for each set of $\eta^\circ(0)$, there is a corresponding T. Conditions that give max. T correspond to a time optimal solution.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \dot{x}_1 \end{bmatrix} .$$

F(x) and the target are defined in the same forms as shown in example (a). Figure 2 shows a set of typical phase coordinate trajectories with $\beta = 0.5$. Note that for the case of $x(0) = \begin{bmatrix} 2.0 \\ 0 \end{bmatrix}$, the time optimal trajectory with the bounded phase-coordinate constraint is identical to that without the constraint. For $x(0) = \begin{bmatrix} 1.7 \\ 0 \end{bmatrix}$, the trajectory which supposedly corresponds to a time optimal solution does not pass through the origin. The trajectory which ends at the origin, however, is not only non-time optimal, but also corresponding to an improper sign of F(x). The latter violates the requirement of convexity of F(x) in the derivation of the computational method.

EXPLANATION

The fact that the computational method could not be implemented directly can be explained by an analysis of a numerical example.

Consider the pure inertia system as given in the previous section. Since $x^0(0) = x_2(0) = 0$ and $\eta^0(0) < 0$, then

$$\hat{f}(0) = \hat{f}|_{t=0} = 0.5|\eta^0(0)| + 1.5\eta_1(0) < 0$$

implies $\eta_1(0) < -|\eta^0(0)|/3 < 0$. Since $\dot{\eta}_1 = 0$, hence $\eta_1(t)$ is a negative constant.

When $|x_2| \leq 0.5$, the function $F(x) = 0$ and $\frac{\partial F'(x)}{\partial x} = 0$ so that $\dot{\eta}_2 = -\eta_1 > 0$. Since $u(t) = \text{sgn}\, \eta_2(t)$, hence $\eta_2(0) < 0$

for otherwise the resulting $u(0) = +1$ would steer the system away from the origin. Thus

$\eta_2(0)$ = negative constant and

$\dot{\eta}_2(0)$ = positive constant.

Since $x_2^*(t) = -t$ for $u(t) = -1$, the trajectory will arrive at the phase coordinate boundary $x_2 = -0.5$ at $t = 0.5$. This is illustrated in Figure 3. Note that $\eta_2(t)$ cannot change sign for $t < 0.5$. If it would, then the switch of the control occurred too early so that the trajectory would not be able to pass through the origin without crossing the other boundary $x_2 = 0.5$ and followed by at least two more switches of control.

Let $\eta_2(t_1) = 0$ at some $t_1 > 0.5$. Then $\eta_2(t)$ is a parabola bending downward for $0.5 \leq t \leq t_1$ since $F(x(t)) = 0.5[x_2(t)+0.5]^2$ $= 0.5[-t+0.5]^2$,

$$\int_{0.5}^{t} \frac{\partial F}{\partial x_2}\, dt = -0.5[-t+0.5]^2$$

and $\eta_2(t) = |\eta_1(0)| - 0.5|\eta^\circ(0)|\cdot[-t+0.5]^2$ for $0.5 \leq t \leq t_1$. For $t = t_1 + \epsilon$, where ϵ is a small quantity, $\eta_2(t) > 0$ and hence $u(t) = +1$. Since $x_2(t) = +t$ for $u(t) = +1$,

$$\int_{t_1}^{t} \frac{\partial F}{\partial x_2}\, dt = +0.5[t + 0.5]^2 - 0.5[t_1+0.5]^2$$

and $\eta_2(t) = |\eta_1(0)| + 0.5\ |\eta^\circ(0)|\cdot\{[t + 0.5]^2 - [t_1+0.5]^2-[-t_1+0.5]^2\}$ which indicates that $\eta_2(t)$ is now a parabola bending upwards for $t > t_1$. Since $\eta_2(t)$ cannot change sign thereafter, hence the trajectory will not pass through the origin unless the switch of the control occurs on the switching curve and in which case the trajectory with the bounded phase-coordinate constraint is identical to that without the constraint.

CONCLUSION

From the above observation, it is concluded that the proposed method [Ref. 1] cannot be directly implemented. It is reasonable to conjecture, however, the method is valid if the parabolic portion of η_2 is tangent to the horizontal axis (Figure 4) such that $\eta_2(t) = \dot{\eta}_2(t) = 0$ for $t_1 \leq t \leq t_2$ and $t_1 < t_2$. But this leaves $u(t) = \text{sgn}\, \eta_2(t)$ undefined on $t \in [t_1, t_2]$ which is equivalent to the introduction of a segment of singular arc. For this case, $\hat{f}$ cannot be readily computed and hence the modified Neustadt's algorithm [Ref. 2] does not apply. Further study of the behavior of singular arc is therefore recommended.

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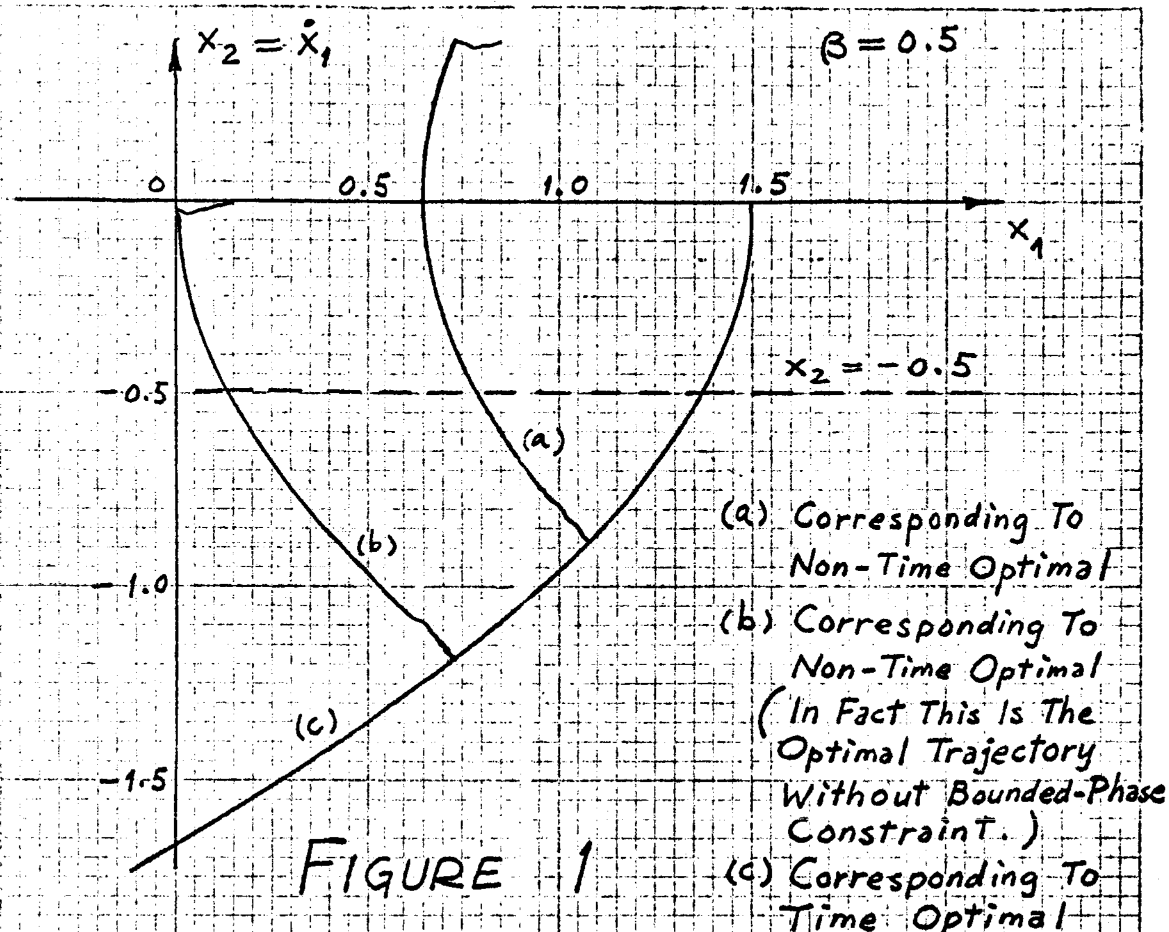

FIGURE 1

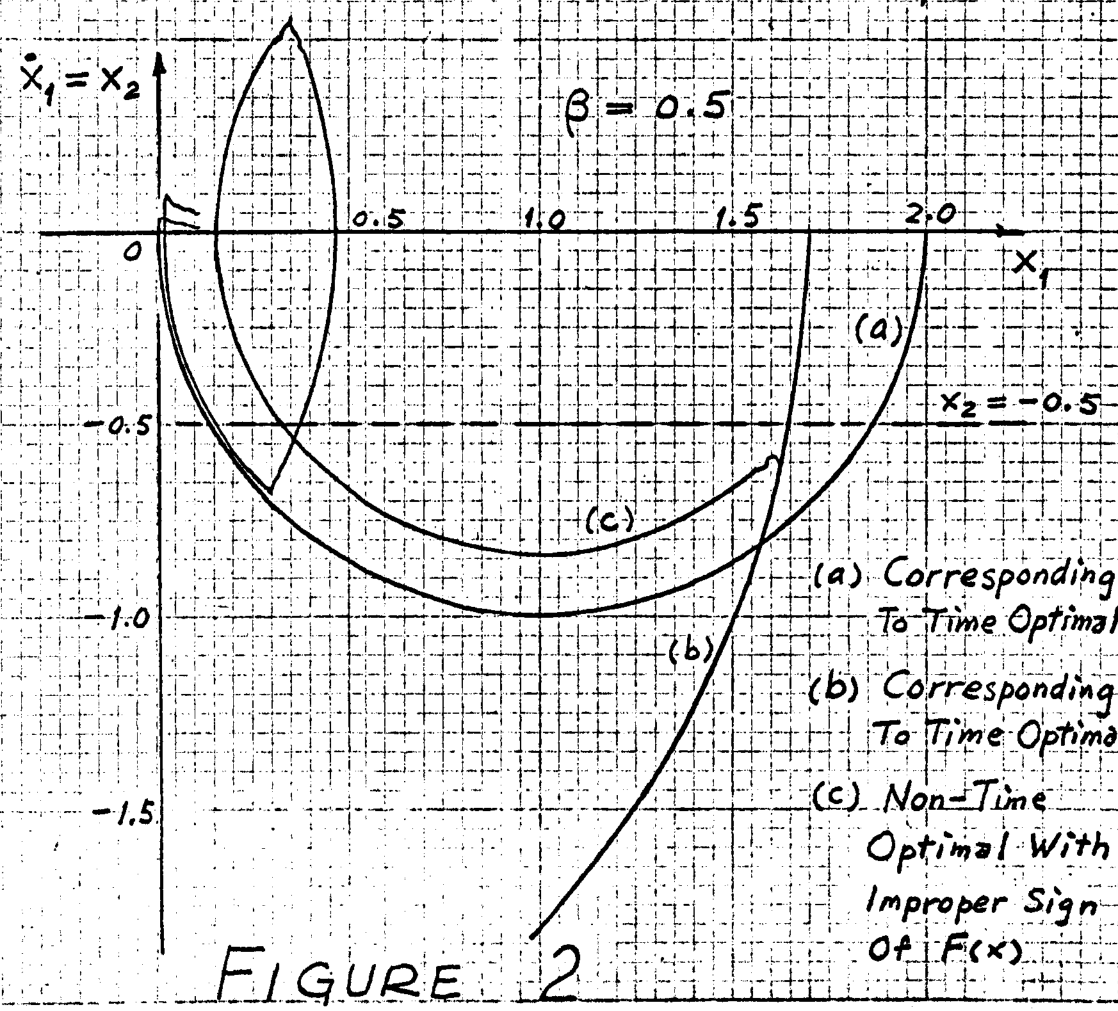

FIGURE 2

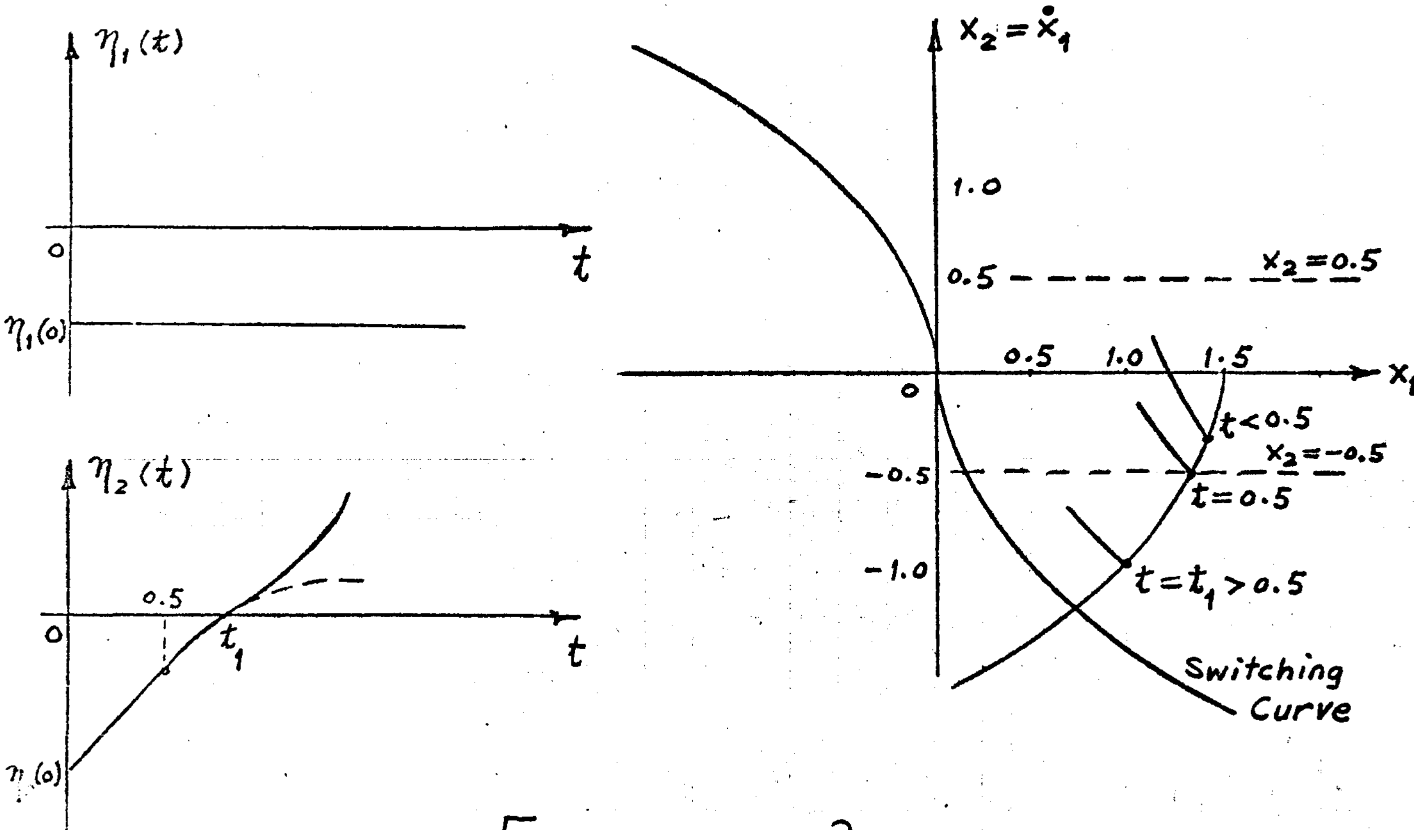

FIGURE 3

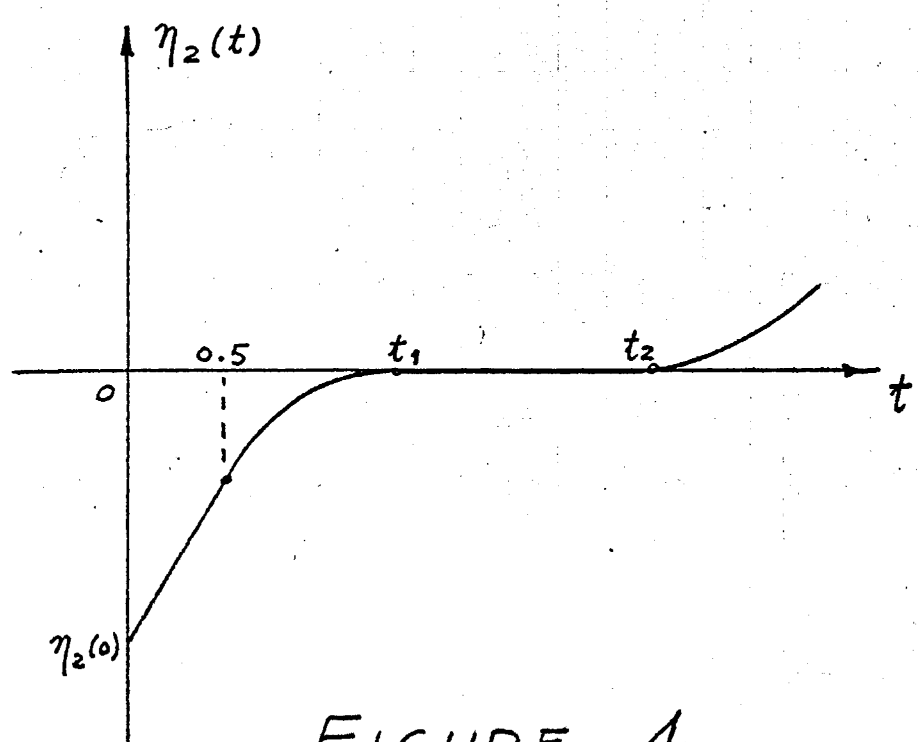

FIGURE 4

APPENDIX C

A SUMMARY OF BOUNDED PHASE-COORDINATE CONTROL PROBLEMS WITH INTEGRAL COST

by

E. B. Lee

A SUMMARY OF BOUNDED PHASE-COORDINATE CONTROL PROBLEMS WITH INTEGRAL COST

By E. B. Lee

In section 4) of Appendix D a short discussion of bounded phase-coordinate problems is given. The motivation for this was the use of certain sufficiency conditions and the usual method of handling bounded phase-coordinate problems by soft constraints introduced through a penalty function. It was later realized that the method used to handle the bounded phase-coordinate time-optimal problem in that appendix was a different method, involving the use of a transversality condition. The use of the transversality condition to obtain a soft constraint appears to be a new way of handling this problem and has not been completely developed and evaluated. It is the purpose of this note to indicate the extent to which the theory for the integral cost criterion can be developed along the lines of the previous theory for time-optimal control with the bounded phase-coordinate.

As in section 4 of Appendix D, consider the linear control process

$\mathcal{L}$ $\quad \dot{x} = A(t)x + B(t)u, \quad x(t_0) = x_0, \ u \in \Omega$, satisfying conditions of section 2 of Appendix D.

The cost functional of control is

$$C(u) = g(x(t_1)) + \int_{t_0}^{t_1} [f^0(x(t),t)]dt,$$

where t_1 is a fixed time $>t_0$ and the real functions $f^{o}(x,t)$ and $h^{o}(x,t)$ are continuously differentiable and $f^{o}(x,t)$ is a convex function of x for each t. g is assumed to be convex in x.

The problem of optimal control is to choose an admissible controller u(t) on $[t_0,t_1]$ so that the response of $\mathcal{L}$ moves from x_0 at t_0 to a target set $G \subset R^n$ at t_1, and minimizes C(u) with the entire response $x_u(t)$ contained in the closed convex restraint set Λ.

As before we introduce the convex differentiable function F(x) satisfying the conditions:

$$F(x) > 0, \quad \text{if } x \notin \Lambda,$$
$$= 0, \quad \text{if } x \in \Lambda.$$

It is at this point that we depart from the previous theory, which was to add F(x) by means of a Lagrange Multiplier λ to the integrand of the integral part of the cost functional and then argue that if λ is sufficiently large the bound on the phase constraint is approximately enforced when C(u) is minimized. Instead we prescribe a bound β and require

$$\int_{t_0}^{t_1} F(x(t))dt \leq \beta.$$

Of course, one way of handling this added inequality is to use the method of Lagrange multipliers, which leads back to the original formulation. We wish to prove existence, as well as give necessary and sufficient conditions, so we will not resort directly to such methods.

Let $\dot{x}^{\circ} = F(x)$,

and $\quad \dot{x}^{n+1} = f^{\circ}(x,t) + h^{\circ}(u,t)$,

with $\quad x^{\circ}(t_o) = 0 = x^{n+1}(t_o)$. We augment the system $\mathcal{L}$ by adding these two equations obtaining the system

$$\tilde{S} \quad \begin{aligned} \dot{x}^{\circ} &= F(x) \\ \dot{x} &= A(t)x + B(t)u \\ \dot{x}^{n+1} &= f^{\circ}(x,t) + h^{\circ}(u,t) \end{aligned}$$

with initial data $\tilde{x}_o = \tilde{x}(t_o) = (x^{\circ}(t_o),\ x(t_o),\ x^{n+1}(t_o)) =$
$= (0,x_o,0)$, $\tilde{x} = (x^{\circ}, x,\ x^{n+1})$.

The soft bound problem is to find an admissible steering function $u(t)\subset\Omega$ on $[t_o,t_1]$ steering $\tilde{x}(t)$ from $\tilde{x}_o$ to the target set $\tilde{G} = \{x^{\circ},x,x^{n+1} \mid 0 \leq x^{\circ} \leq \beta,\ x\in G,\ 0 \leq x^{n+1} < \infty\}$ with minimum $x^{n+1}(t_1) + g(x(t_1))$.

Define the set of attainability $\tilde{K}(t_1)$, in variables (x°,x,x^{n+1}) to be the collection of end point $\tilde{x}(t_1)$ of responses $\tilde{x}(t)$ of $\tilde{S}$ corresponding to all admissible controllers $u(t)$ on $[t_o,t_1]$ with $\tilde{x}(t_o) = \tilde{x}_o$. An admissible controller $u(t)$ is a measurable controller belonging to the compact set Ω.

An adjoint response corresponding to an admissible controller $u(t)$ on $[t_o,t_1]$ is n+2 row vector satisfying the differential system

$$\eta_o \equiv \text{constant} \leq 0$$

$$\dot{\eta} = -\eta\, A(t) - \eta_o \frac{\partial F}{\partial x}\, (x(t)' - \eta_{n+1} \frac{\partial f^{n+1}}{\partial x}\, (x(t))'$$

$\dot{\eta}_{n+1} \equiv$ constant ≤ 0, on $[t_o,t_1]$ where $x(t)$ is the response of $\mathcal{L}$ corresponding to the controller $u(t)$.

Define an admissible controller $u(t)$ on $[t_o, t_1]$ to be extremal if the response $\tilde{x}(t)$ of $\tilde{S}$ corresponding to $u(t)$ has an end point $\tilde{x}(t_1)$ contained in the lower boundary (in both x° and x^{n+1}) of $\tilde{\tilde{K}}(t_1)$. Further, define an admissible controller $u(t)$ on $[t_o, t_1]$ to be a maximal controller in case there exists a nonvanishing adjoint response $\tilde{\eta}(t)$ such that

$$\tilde{\eta}(t)\, B(t)u(t) = \max_{u \in \Omega} \{\tilde{\eta}(t)B(t)u\},$$

a.e. on $[t_o, t_1]$.

It has been established that $\tilde{K}(t_1)$ is a compact subset of R^{n+2}, and so we are assured that optimum controllers exist. It appears that the lower boundary in x° and x^{n+1} is a convex surface so that the extremal and maximal controllers are the same, giving us a way of choosing optimim controllers. Further, it is expected that theorems similar to the remaining theorems of Appendix D will be obtained.

APPENDIX D

AN APPROXIMATION TO LINEAR BOUNDED PHASE COORDINATE CONTROL PROBLEMS

by

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(Revised)

(REVISED)

AN APPROXIMATION TO LINEAR BOUNDED PHASE COORDINATE CONTROL PROBLEMS*

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1. Introduction

In many control problems both restraints on the magnitudes of the control variables and various system variables may occur. Certain results [1,2,7] are available for the determination of optimal controllers for some classes of linear and nonlinear systems involving such restraints. These results take the form of necessary or sufficient conditions for optimal control but not both, and are therefore only a partial solution to even the theoretical problem, leaving much to be desired in the way of a practical solution. To use the necessary or sufficient conditions for synthesizing an optimal controller it is necessary to solve a two-point boundary value problem in terms of a number of free parameters and multipliers where the number of parameters is not even known as well as certain jump conditions [2,7]. A backing out procedure [9] is also available if one is interested in flooding the domain of controllability with responses and then keeping track (storing) of the corresponding control magnitude for each such point.

We here offer a procedure which has several advantages over the above schemes, but is only an approximate solution. Its main advantage is that no discontinuities will be encountered in the adjoint solution which determines the optimum controller and therefore the resulting two point boundary value problem may be more readily solved. The results provide both necessary and sufficient conditions, as well as existence,

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for the approximate problem.

The analysis is limited to linear control processes as described by the differential system

$$\mathcal{L})\ \dot{x} = A(t)x + B(t)u(t).$$

The coefficient matrices $A(t)$ and $B(t)$ are composed of known continuous functions on the time interval $[t_0, t_1]$. The controller $u(t)$ is to be chosen from a set $\Omega: |u^j| \leq 1;\ j = 1,2,\ldots m$, so as to steer the response, $x_u(t)$, of $\mathcal{L})$ from an initial point x_o at time t_o to a prescribed compact target set $\tilde{G} \subset R^n$ and it is required that $x_u(t)$ remain within a given constraint set, Λ, during its entire response. Here R^n is the n dimensional real number space.

The problem of time optimal control, as considered in the next section, is to find a controller $u(t)$ which steers $x_u(t)$ from x_o to $\tilde{G} \subset \Lambda$ in minimun time, that is, minimizes $C(u) = t_1 - t_o$ with $x(t_1) \epsilon\ \tilde{G}$ and $x_u(t)\ \epsilon\ \Lambda,\ t_o \leq t \leq t_1$.' Later, in section 4, we discuss other optimum control cost functionals.

There are certain difficulties involved when one directly solves for this optimum controller. We shall therefore be content with solving the following apparently simpler problem: Find that controller $u(t)$ with graph in Ω which steers $x_u(t)$ from x_o at t_o to $\tilde{G}$ at t_1 with $x_u^o(t_1) \leq \beta$ and $t_1 - t_o$ a minimum. $x_u^o(t)$ is defined below.

It is assumed that Λ is a closed convex set, (for convenience we could even let $\Lambda = \{x | x'H\,x \leq c\}$, where H is a positive semi-definite matrix and c = constant > 0.) Let $F(x)$ be a convex continuous differentiable function which is such that

$$\begin{aligned} F(x) &\neq 0 \qquad \text{if } x \not\in \Lambda \\ &= 0 \qquad \text{if } x \in \Lambda \end{aligned}$$

Then define†

$$x_u^\circ(t_1) = \int_{t_o}^{t_1} F(x_u(t))dt.$$

$x_u^\circ(t_1)$ essentially measures the excursions of the response $x_u(t)$ to a controller $u(t)$ outside of the region Λ during the time interval $[t_o,t_1]$. By keeping $x_u^\circ(t_1)$ small the response $x_u(t)$ is restricted to stay close to or within Λ. The above minimum time optimal control problem is approximately solved by finding a controller which steers $\hat{x}_u(t) = (x_u^\circ(t)x_u(t))$ from $(0,x_o)$ to $G = \{x^\circ,x | x \in \tilde{G}, 0 \leq x^\circ \leq \beta\}$ in the minimum time interval $t_1 - t_o$ if $\beta > 0$ is sufficiently small.

In the next section we give necessary and sufficient conditions for this approximation problem using the time optimal criterion. Section 3 contains an example and section 4 is a discussion of the approximation problem for other cost functionals.

2. The necessary and sufficient conditions for the approximate linear time optimal problems

We augment the system Σ by considering the equation system

†There is, of course, some question as to whether such a function $F(x)$ exists for an arbitrary convex set Λ contained in R^n. We now cite an example which shows that there are such functions in a number of interesting cases. Suppose $\Lambda = \{x^1, x^2,\ldots x^n | |x^2| \leq 1\}$. Then pick $F(x)$

$$\begin{aligned} &= 1/2(x^2 - 1)^2 && \text{if } x^2 \geq 1 \\ &= 0 && \text{if } |x^2| \leq 1 \\ &= 1/2(x^2 + 1)^2 && \text{if } x^2 \leq -1 \end{aligned}$$

Thus if only one coordinate (or a linear combination) is restricted the problem is easily handled as in the example, where $F(x)$ is continuous and has continuous partial derivatives. Other Λ's can be approximately handled as in the example.

$$\hat{\Sigma})\quad \dot{x}^\circ = F(x)$$

$$\dot{x} = A(t)x + B(t)\,u(t)$$

obtained from Σ) by adding the equation for $\dot{x}^\circ$ with $x^\circ(t_o) = 0$. Here $A(t)$, $B(t)$ are bounded and continuous on $[t_o, t_1]$ and $F(x)$ is a convex function with $F(x) = 0$ for $x \in \Lambda$. $\frac{\partial F}{\partial x}(x)$ is assumed to exist and be continuous everywhere.

The set of attainability $\hat{K}(t_1) \subset R^{n+1}$ is the collection of end points $\hat{x}_u(t_1)$ of responses $\hat{x}_u(t) = (x_u^\circ(t), x_u(t))$ of $\hat{\Sigma}$ which initiate at $(0, x_o)$ at time t_o corresponding to all (Lebsegue) measurable controllers $u(t)$ which are such that $|u^j(t)| \leq 1$ on $[t_o, t_1]$, for $j = 1,2\ldots,m$. (Such controllers are referred to as admissible controllers.)

In the following theorems we establish various properties for $\hat{K}(t_1)$ and $\partial\hat{K}(t_1)$ as required in synthesizing optimal controllers.

Theorem 1 Consider the above system $\hat{\Sigma}$) with initial point $\hat{x}_o$, restraint set Ω, and set of attainability $\hat{K}(t_1)$. Then $\hat{K}(t_1)$ is a nonempty compact subset of R^{n+1} in variables (x°, x) with convex lower surface (as defined below) for each $t_o \leq t_1 < \infty$.

Proof $\hat{K}(t_1)$ is nonempty since any measurable controller $u(t) \subset \Omega$ gives rise to an end point $\hat{x}_u(t_1) \in \hat{K}(t_1)$. $\hat{K}(t_1)$ is compact because the system $\hat{\Sigma}$) satisfies the hypothesis of the existence theorems of references 6, and 8.

The lower surface[†] of $\hat{K}(t)$ is where exterior normal vectors $\hat{\eta}$ to $\hat{K}(t)$ at points of $\partial\hat{K}(t)$ have their first component $\eta_o \leq 0$. We now show that if $\hat{x}_1$ and $\hat{x}_2$ are points of $\hat{K}(t_1)$ then the point $\hat{y} = \lambda\hat{x}_1 + (1-\lambda)\hat{x}_2 = (y^\circ, y)$, $0 \leq \lambda \leq 1$, is such that

$$y = x_{\bar{u}}(t_1)$$

and

$$y^\circ \geq x^\circ_{\bar{u}}(t_1),$$

where $\bar{u}(t) = \lambda\, u_1(t) + (1-\lambda)\, u_2(t)$ and $u_1(t)$ and $u_2(t)$ are such that $\hat{x}_{u_1}(t_1) = \hat{x}_1$ and $= \hat{x}_{u_2}(t_1) = \hat{x}_2$. The convexity of the lower surface of $\hat{K}(t_1)$ then follows because in order for it to be nonconvex it is necessary that there exist two points $\hat{x}_1$, $\hat{x}_2$ on this lower boundary, with the property that the point $\lambda\, \hat{x}_1 + (1-\lambda)\, \hat{x}_2$ is below the set $\hat{K}(t_1)$ for some $0 < \lambda < 1$, which will then be impossible.

With $\bar{u}(t) = \lambda\, u_1(t) + (1-\lambda)\, u_2(t)$ we find that

$$\begin{aligned} x_{\bar{u}}(t_1) &= \Phi(t_1)x_o + \int_{t_o}^{t_1} \Phi(t_1)\Phi^{-1}(s)B(s)\bar{u}(s)ds \\ &= \lambda \left[\Phi(t_1)x_o + \int_{t_o}^{t_1} \Phi(t_1)\Phi^{-1}(s)B(s)u_1(s)ds\right] \\ &+ (1-\lambda) \left[\Phi(t_1)x_o + \int_{t_o}^{t_1} \Phi(t_1)\Phi^{-1}(s)B(s)u_2(s)ds\right] \\ &= \lambda\, x_{u_1}(t_1) + (1-\lambda)\, x_{u_2}(t_1) = \\ &= \lambda\, x_1 + (1-\lambda)\, x_2 = y \end{aligned}$$

where $\Phi(t)$ is the fundamental solution matrix of $\mathfrak{L}$ with $\Phi(t_o) = I$. We also calculate

$$x^\circ_{\bar{u}}(t_1) = \int_{t_o}^{t_1} F(x_{\bar{u}}(t))dt$$

and $\lambda\, x^{\circ}_{u_1}(t_1) + (1-\lambda)\, x^{\circ}_{u_2}(t_1)$ for comparison. Since $F(x)$ is a convex function of $\hat{x}$ it follows that for $0 \le \lambda \le 1$,

$$F(x_{\bar{u}}(t)) = F(\lambda\, x_{u_1}(t) + (1-\lambda)x_{u_2}(t)) \le \lambda\, F(x_{u_1}(t)) + (1-\lambda)\, F(x_{u_2}(t))$$

and so

$$x^{\circ}_{\bar{u}}(t_1) = \int_{t_o}^{t_1} F(x_{\bar{u}}(t))dt = \int_{t_o}^{t_1} F(\lambda\, x_{u_1}(t) + (1-\lambda)\, x_{u_2}(t))dt$$

$$\le \lambda \int_{t_o}^{t_1} F(x_{u_1}(t))dt + \int_{t_o}^{t_1} (1-\lambda)F(x_{u_2}(t))dt = y^{\circ}.$$

Q.E.D.

We will now consider those controllers $u(t)$ on $[t_o,t_1]$ which steer $\hat{x}_u(t)$ from $\hat{x}_o$ at t_o to points $\hat{x}_1$ contained in the lower boundary of $\hat{K}(t_1)$ (written $\partial\hat{K}^-(t_1)$). Such controllers will be called extremal and they will play a significant part in the selection of optimal controllers.

Let $u(t) \in \Omega$ on $t_o \le t \le t_1$ be an admissible controller for the convex control process

$$\hat{\Sigma})\quad \dot{x}^{\circ} = F(x)$$

$$\dot{x} = A(t)\, x + B(t)u(t)$$

with initial point $\hat{x}_o = (0,x_o)$ at t_o. If the corresponding response $\hat{x}_u(t)$ has an end point $\hat{x}(t_1)\epsilon\partial K^-(t_1)$, then $u(t)$ is called an extremal control and $\hat{x}_u(t)$ an extremal response on $[t_o,t_1]$.

The adjoint response $\hat{\eta}(t) = (\eta_0(t), \eta(t))$ corresponding to a controller $u(t)$ is a row $n+1$ vector satisfying the f differential system

$$\dot{\eta} = -\eta A(t) - \eta_0 \frac{\partial F'}{\partial x}(x_u(t))$$

$$\eta_0 = \text{constant} \lessgtr 0.$$

where $x_u(t)$ is the response of $\mathcal{L})$ corresponding to the controller $u(t)$. Define $u(t)$ on $[t_0,t_1]$ to be a maximal controller in case there exists a nonvanishing adjoint response $\hat{\eta}(t)$, $\eta_0 \leq 0$, so that $\eta(t)B(t)u(t) = \underset{u\in\Omega}{\text{Max}} \{\eta(t)B(t)u\}$ a.e. on $[t_0,t_1]$.

In the following theorem 2 it is shown that extremal and maximal controllers are the same.

<u>Theorem 2 Consider the convex control process</u>†:

$$\hat{\mathcal{L}}) \quad \dot{x}^0 = F(x)$$

$$\dot{x} = A(t)x + B(t)u(t)$$

<u>with initial point $\hat{x}_0 = (0,x_0)$ at time t_0. An admissible controller $u(t) \subset \Omega$ on $[t_0,t_1]$ is extremal for $\hat{\mathcal{L}}$ if and only if it is a maximal controller, that is, if and only if there is a nonvanishing adjoint response $\hat{\eta}(t)$ of</u>

$$\dot{\eta} = -\eta A(t) - \eta_0 \frac{\partial F'}{\partial x}(x_u(t))$$

$$\eta_0 = \text{constant} \lessgtr 0$$

†The necessary portion of this theorem follows from L. S. Pontryagin's Maximum Principle (7). For completeness the simple arguments to establish the necessary part are presented.

so that

$$\eta(t)B(t)u(t) = \underset{u\epsilon\Omega}{\text{Max}} \{\eta(t)B(t)u\} \text{ almost always on } [t_o,t_1].$$

Proof Assume $u(t)$ on $[t_o,t_1]$ is extremal and so steers $\hat{x}(t)$ from $(0,x_o)$ at t_o to $\hat{x}_1 \epsilon\, \partial\hat{K}^-(t_1)$ at t_1. Choose $\hat{\eta}(t_1) = (\eta_o,\eta(t_1))$ to be a nonzero vector normal to π directed into the halfspace defined by π which does not meet $\hat{K}(t_1)$. Note $\eta_o \lessgtr 0$. Then let $\hat{\eta}(t)$ with $\hat{\eta}(t_1)$ as above be the response of the adjoint equation corresponding to the controller $u(t)$.

The controller [+] $\bar{u}(t) = \text{sgn}\{\eta(t)B(t)\}$ defined for $t \epsilon [t_o,t_1]$ is admissible and

$$\eta(t)B(t)\bar{u}(t) = \underset{u\epsilon\Omega}{\text{Max}} \{\eta(t)B(t)u\}$$

on $[t_o,t_1]$.

Let τ_ϵ be an interval of total length $\epsilon > 0$ contained in $\mathcal{J} = [t_o,t_1]$ whereon

$$\delta + \eta(t)B(t)u(t) < \underset{u\epsilon\Omega}{\text{Max}} \{\eta(t)B(t)u\} \text{ for some } \delta > 0.$$

For given $\delta > 0$ consider the modified controller

$$u_\epsilon(t) = u(t) \text{ on } \mathcal{J} - \tau_\epsilon$$
$$= \bar{u}(t) \text{ on } \tau_\epsilon,$$

+ $\text{sgn}\,\{\} = -1$ if $\{\} < 0$
$= 0$ if $\{\} = 0$
$= +1$ if $\{\} > 0$

and calculate

$$\frac{d\hat{\eta}(t)\hat{x}_\epsilon}{dt} = \dot{\hat{\eta}}\hat{x}_\epsilon + \eta\dot{\hat{x}}_\epsilon$$

and

$\frac{d\hat{\eta}(t)\hat{x}}{dt} = \dot{\hat{\eta}}\hat{x} + \hat{\eta}\dot{\hat{x}}$, where $\hat{x}_\epsilon$ refers to a response of $\hat{\mathfrak{L}}$ corresponding to the modified controller $u_\epsilon(t)$.

Integration from t_o to t_1 yields

$$\hat{\eta}((t)\ \hat{x}_\epsilon(t_1) - \hat{\eta}(t)\hat{x}_\epsilon(t_o) = \int_{t_o}^{t_1} \left[-\eta\ A(t) + \frac{\partial F}{\partial x}(x(t))\right]x_\epsilon(t)$$

$$+ \int_{t_o}^{t_1} \eta(t)\left[A(t)x_\epsilon(t) + B(t)u(t)\right] - F(x_\epsilon(t))dt$$

and

$$\hat{\eta}(t_1)\hat{x}(t_1) - \hat{\eta}(t_o)\ \hat{x}(t_o) = \int_{t_o}^{t_1} \left\{\left[-\eta\ A(t) + \frac{\partial F}{\partial x}\ (x(t))\right]x(t)\right.$$

$$\left. + \eta(t)\left[A(t)x(t) + B(t)u(t)\right] - F(x(t))\right\}dt \text{ for } \eta_o = -1.$$ The case

Combining terms and using the assumed continuity for F and $\frac{\partial F}{\partial x}$ we easily find that

$\hat{\eta}(t_1)\hat{x}_\epsilon(t_1) - \hat{\eta}(t_1)\hat{x}(t_1) \geq \delta\ \epsilon + o(\epsilon)$ for ϵ sufficiently small where $o(\epsilon)$ corresponds to terms of higher than first order in ϵ, and therefore for ϵ sufficiently small

$\hat{\eta}(t)x_\epsilon(t_1) - \hat{\eta}(t_1)\hat{x}(t_1) > 0$, contradicting the construction of $\hat{\eta}(t_1)$ as the outward normal to $\hat{K}(t_1)$ at $\hat{x}_1$.

Hence there exists no such interval τ_ϵ, so

$$\eta(t)B(t)u(t) = \operatorname*{Max}_{u\epsilon\Omega} \eta(t)B(t)u \text{ almost everywhere on } \mathcal{S}.$$

Conversely, assume that u(t) and corresponding response $\hat{\eta}(t) \neq 0$ are such that

$$\eta(t)B(t)u(t) = \operatorname*{Max}_{u\epsilon\Omega} \eta(t)Bu$$

a.e. on $\mathcal{S}$ with $\eta_o \lessgtr 0$. Let $\bar{u}(t)$ be any controller in Ω with corresponding response $x_{\bar{u}}(t)$. If we calculate

$$\frac{d\hat{\eta}\hat{x}_u}{dt} \text{ and } \frac{d\hat{\eta}\hat{x}_{\bar{u}}}{dt} \text{ as above,}$$

and then integrate from t_o to t_1 using the assumed convexity of F(x) we find that

$$\hat{\eta}(t_1)\ \hat{x}_u(t_1) \geq \hat{\eta}(t_1)\ \hat{x}_{\bar{u}}(t_1) = \hat{\eta}(t_1)\hat{w}$$

where $\hat{w}$ is any point of $\hat{K}(t_1)$. Since $|\hat{\eta}(t_1)| \neq 0$, and $\eta_o \leq 0$, the above inequality implies that $\hat{x}_u(t_1)$ is contained in the lower boundary of the compact set $\hat{K}(t_1)$ with convex lower boundary and hence u(t) is extremal QED.

Theorem 2 indicates that to stay at a lower boundary point we must continuously steer maximally in the direction of the vector $\hat{\eta}(t)$. This remark is summarized as a corollary.

Corollary 2.1 Let u(t) on $[t_o, t_1]$ be an extremal controller <u>for $\mathcal{L}$, with corresponding response $\hat{x}_u(t)$ and adjoint response $\hat{\eta}(t)$ so that,</u>

$$\eta(t)B(t)u(t) = \operatorname*{Max}_{u\epsilon\Omega} \eta(t)B(t)u$$

a.e. on $[t_0,t_1]$. Then on each subinterval $[t_0\tau]\subset[t_0,t_1]$, $u(t)$ is also an extremal controller with $\hat{x}_u(\tau)\epsilon\partial\hat{K}(\tau)$. Moreover $\hat{\eta}(\tau)$ is an exterior normal to $\hat{K}(\tau)$ at $\hat{x}(\tau)$.

Proof Replace t_1 by τ in the proof of theorem 2 to obtain that

$$\hat{\eta}(\tau)\ \hat{x}_u(\tau) \geq \hat{\eta}(\tau)\ \hat{x}_{\bar{u}}(\tau) = \hat{\eta}(\tau)\ \hat{w}(\tau)$$

for all $\hat{w}(\tau)$ in $\hat{K}(\tau)$. From this inequality the conclusion of the corollary can be drawn.

We next show that the set of attainability $\hat{K}(t_1)$ depends continuously on the parameter t_1.

Define the distance between a point p and a compact set $G_1\subset R^n$ to be

$$d(p,G_1) = \underset{g\epsilon G_1}{\mathrm{Min}}\ |p-g|$$

and define the distance between two compact sets G_1, and $G_2\subset R^n$ to be

$$d(G_2,G_2) = \mathrm{Max}\left\{\underset{p_1\epsilon G_1}{\mathrm{Max}}\ d(p_1,G_2),\ \underset{p_2\epsilon G_2}{\mathrm{Max}}\ d(p_2,G_1)\right\}.$$ Here

$$|p| = \sum_{i=1}^{n} |p^i|.$$

The set $\hat{K}(t_2)\subset R^{n+1}$ varies continuously with t_2 if given an $\epsilon > 0$ there exists a $\delta > 0$ so that for $|t_2-t_1| < \delta$,

$$d(\hat{K}(t_1),\ \hat{K}(t_2)) < \epsilon$$

Lemma 1 Consider the system $\hat{\Sigma}$ as above with attainable set $\hat{K}(t_1) \subset R^{n+1}$. Then $\hat{K}(t_1)$ varies continuously with $t_1 < \infty$.

Proof We need only show that each point $\hat{x}(t_1)$ of $\hat{K}(t_1)$ is close to some point $\hat{x}(t_2)$ of $\hat{K}(t_2)$ and conversely. That is, we need show that given $\epsilon > 0$ there exists a $\delta > 0$ so that when $|t_1 - t_2| < \delta$ there exists $\hat{x}(t_1) \in \hat{K}(t_1)$ such that $|x(t_1) - x(t_2)| < \epsilon$ for each $\hat{x}(t_2) \in \hat{K}(t_2)$ and conversely.

Let $u_1(t)$ be an admissible controller on $[t_o, t_1+1]$ and $\hat{x}_1(t)$ the corresponding response. For $t_1 \leq t_2 \leq t_1 + 1$ calculate

$$x_1^o(t_2) - x_1^o(t_1) = \int_{t_o}^{t_2} F(x_1(t))dt - \int_{t_o}^{t_1} F(x_1(t))dt$$

and

$$\begin{aligned} x_1(t_2) - x_1(t_1) = {} & \Phi(t_2) \int_{t_o}^{t_2} \Phi(s)^{-1} B(s)u_1(s)ds \\ & - \Phi(t_2) \int_{t_o}^{t_1} \Phi(s)^{-1}[B(s)u_1(s)]ds \\ & + [\Phi(t_2) - \Phi(t_1)] \left[\int_{t_o}^{t_1} \Phi(s)^{-1} B(s)u_1(s)ds\right]. \end{aligned}$$

So

$$x_1^o(t_2) - x_1^o(t_1) = \int_{t_1}^{t_2} F(x_1(t))dt$$

and

$$x_1(t_2) - x_1(t_1) = \Phi(t_2) \int_{t_1}^{t_2} \Phi(s)^{-1}u_1(s)ds$$

$$+ \left[\Phi(t_2) - \Phi(t_1)\right]\left[\int_{t_o}^{t_1} \Phi(s)^{-1} B(s)u_1(s)ds\right]$$

Since $A(t)$ is bounded and continuous on $[t_o,t_1+1]$ so is $\Phi(t)$ and therefore there exists a constant C_1 so that

$$|\Phi(t)| < C_1$$

and

$$|\Phi(t)^{-1}| < C_1 \text{ on } [t_o,t_1+1].$$

Also since $B(s)$ has bounded continuous elements $b_j^i(t)$ and $u_1(t)$ is bounded amd measurable there exists the constant C_2 so that

$|\int_{t_o}^{t_1} \Phi(s)^{-1} B(s)u_1(s)ds| < C_2$. Integration is a continu8us operation, therefore, given an $\epsilon > 0$ there exists a $\delta > 0$ so that

$$|\int_{t_1}^{t} F(x_1(t))dt| < \frac{\epsilon}{3},$$

$$|\int_{t_1}^{t} \Phi(s)^{-1} B(s)u_1(s)ds| < \frac{\epsilon}{3C_2}$$

for $|t-t_1| < \delta < 1$.

Hence

$$|\hat{x}_1(t_2) - \hat{x}_1(t_1)| < \frac{\epsilon}{3} + C_1 \frac{\epsilon}{3C_1} + \frac{\epsilon}{3C_2} C_2 = \epsilon$$

for $|t_2 - t_1| < \delta < 1$.

The other way we consider $u_1(t) = u(t)$ on $[t_0, t_1]$ where $u(t)$ steers to $\hat{x}(t_1)$ and extend it to $[t_0, t_1+1]$ by letting $u_1(t) = u(t_1)$ for $t \in [t_1, t_1+1]$. The above calculation is then repeated to find $|\hat{x}(t_2) - \hat{x}(t_1)| < \epsilon$ for $|t_2 - t_1| < \delta < 1$ and so $\hat{K}(t_1)$ varies continuously with t_1.

Theorem 3 Consider the system $\hat{\Sigma}$ as above with initial data $\hat{x}_0 = (0, x_0)$, compact restraint set Ω, and set of attainability $\hat{K}(t_1)$. Let the target set $G = \{x^0, x \mid 0 \leq x^0 \leq \beta,\ x \in \tilde{G}\}$ where $\beta > 0$ is a constant and $\tilde{G}$ is a compact set of R^n. Suppose G meets the interior of $\hat{K}(t_1)$, then there is a $\delta > 0$ such that G meets $\hat{K}(t_1)$ for $|t - t_1| < \delta$.

Proof Since G meets the interior of $\hat{K}(t_1)$, there is a point $\hat{p} \in (G \cap \text{Int. } \hat{K}(t_1))$ and a ball neighborhood $N(\hat{p})$ of radius $r > 0$ contained in $\hat{K}(t_1)$. Consider the hyperplane $x^0 = p^0 - r/2$ of R^{n+1} and in this plane pick $n+1$ independent points $\hat{x}_1$, $\hat{x}_2 \ldots \hat{x}_n$, $\hat{x}_{n+1}$ of the boundary of the ball $N(\hat{p})$, all equally spaced. Let $\hat{x}_1(t), \hat{x}_2(t), \ldots \hat{x}_n(t), \hat{x}_{n+1}(t)$ be responses of $\hat{\Sigma}$ with initial data $\hat{x}_0 = (0, x_0)$ and corresponding to controllers $u_1(t), u_2(t), \ldots u_{n+1}(t)$, $t_0 \leq t \leq t_1 + 1$, which are such that $\hat{x}_1(t_1) = \hat{x}_1, \ldots \hat{x}_{n+1}(t_1) = \hat{x}_{n+1}$. Pick $1 > \delta > 0$ so small that for $|t - t_1| \leq \delta$ the points $\hat{x}_1(t)$ lie within spheres of radius $r/10$ of the points $\hat{x}_1 \ldots \hat{x}_{n+1}$. This being possible because of the previous lemma 1.

Consider the convex combination of controllers $u_\lambda(t) = \lambda_1 u_1(t) + \lambda_2 u_2(t) + \ldots \lambda_{n+1} u_{n+1}(t)$, $\lambda_i > 0$, $\Sigma \lambda_i = 1$ (Note $|u_\lambda{}^i| \leq 1$) and the corresponding responses $\hat{x}_\lambda(t)$ of $\hat{\mathcal{S}}$ with initial data $(0, x_o)$. For each fixed t, $|t-t_1| \leq \delta$ these response end points $x_\lambda(t)$ sweep out a surface section $\tilde{S}$ which lies below the plane $x^o = p^o$ by convexity, above or on the plane $x^o = 0$ because of the positive nature of F and intersect the line segment $\{0 \leq x^o \leq p^o, x=p\}$ (see proof of theorem 1). Hence G meets $\hat{K}(t)$ for $|t-t_1| \lessgtr \delta < 1$.

We now consider the problem of existence of optimum controllers.

<u>Theorem 4</u> <u>Consider the system $\hat{\mathcal{S}}$ as above with compact restraint set $\Omega = \{u \mid |u^i| \leq 1, i=1,2\ldots,m\} \subset R^n$, initial point $(0,x_o) \epsilon R^{n+1}$ at time t_o and constant compact target set $G = \{x^o, x \mid 0 \leq x^o \leq \beta, x \epsilon G\}$ for $\beta > 0$. If there exists an admissible controller $u(t) \subset \Omega$ steering $\hat{x}_o$ to G on $t_o \leq t \leq t_1$ then there exists an optimum controller (also admissible) steering $\hat{x}$ to G in minimum time duration $t^* - t_o$.</u>

<u>Proof</u> If $(0,x_o) \epsilon$ G then $t^* = t_o$ and optimum control is not required. So assume $(0,x_o) \not\epsilon$ G and consider the set of attainability $\hat{K}(t_1)$ for $t_1 \geq t_o$. Since there is one controller which steers $(0,x_o)$ to G the set $\hat{K}(t_1)$ meets G for some $t_1 > t_o$. Define t^* to be the greatest lower bound of all times t_1 such that $\hat{K}(t_1)$ meets G. By the continuous dependence of $\hat{K}(t_1)$ on t_1 the set of times for which $K(t_1)$ meets G is a closed set in R^1. Hence t^* is the first time $\hat{K}(t_1)$ meets G and therefore pick as the optimum controller $u^*(t)$, $t_o \leq t \leq t^*$, a controller which steers to

$K(t^*) \cap G$.

The next theorem asserts that for optimum control we need only consider points of the lower boundary of the set of attainability and therefore by theorem 2 extremal controllers. A sufficiency condition is also included.

Theorem 5. Consider the system $\hat{\Sigma}$ as above with compact rectangular restraint set Ω, initial point $(0,x_o)$ at t_o and compact convex target set $G = \{x^o, x | 0 \leq x^o \leq \beta; x \in G; \beta > 0\}$. Let $u^*(t)$ be a minimal time optimal controller steering $\hat{x}^*(t)$ from $\hat{x}_o$ to G. Then $u^*(t)$ is extremal, that is, there exists a nonvanishing adjoint response $\hat{\eta}(t) = (\eta_o, \eta(t))$ with $\eta_o \leq 0$ so that

$$\eta(t)B(t)u^*(t) = \underset{u\in\Omega}{\mathrm{Max}} \{\eta(t)B(t)u\}$$

almost always on $[t_o, t^*]$ with $\hat{\eta}(t^*)$ an outward normal of $\hat{K}(t^*)$ at $x^*(t^*)$ on $\partial\hat{K}(t^*)$ and $\hat{\eta}(t^*)$ satisfies the transversality condition, namely, $\hat{\eta}(t^*)$ is normal to a supporting hyperplane π of G and the set of attainability $\hat{K}(t^*)$ which separates $\hat{K}(t^*)$ from G.

Moreover, if for each point [3] $\bar{x} \in G$ there exists a nonmaximal controller $\bar{u}(t) \subset \Omega$ so that on $\bar{t}_o \leq t < \infty$ the response $x_{\bar{u}}(t)$ initiating at $\bar{x} = x_{\bar{u}}(\bar{t}_o)$ is contained in G, then when $u(t)$ is an admissible extremal controller steering x_o to G by means of a response satisfying the transversality condition it is an optimum controller.

Proof By assumption there exists a controller steering $\hat{x}_o$ to G so G meets $\hat{K}(t^*)$. Suppose G meets the interior of $\hat{K}(t^*)$. This is impossible because then G meets the interior of $\hat{K}(t)$ for $|t-t^*| < \delta$, $\delta > 0$, by theorem 3 and this contradicts the optimality of the controller. Hence ∂G meets $\partial\hat{K}(t^*)$ so that the optimum controller must steer to $\partial\hat{K}(t^*)$. We must show that it steers to a lower boundary point to conclude that it is extremal. This follows at once because $\hat{K}(t)$ always first makes contact with G at a lower boundary point as can be seen by considering how the compact set $\hat{K}(t_1)$ with convex lower surface moves with respect to the set G. Thus if $u^*(t)$ is optimal it is extremal and by theorem 2 there exists the nonvanishing adjoint response $\hat{\eta}(t)$ so that

$$\eta(t)B(t)u^*(t) = \max_{u \in \Omega} \eta(t)B(t)u$$

where $\hat{\eta}(t^*)$ satisfies the transversality condition since G and the lower boundary of $\hat{K}(t^*)$ are convex they can be separated by a supporting hyperplane π and we choose $\hat{\eta}(t^*)$ to be normal to π and directed into the halfspace containing G.

When $u(t)$ is an admissible extremal controller steering $\hat{x}_o$ to G and satisfying the transversality condition it must be an optimum controller if G has the property that through each point $\bar{x} \epsilon G$ there passes a nonmaximal response which remains forever in G. This follows because once G and $\hat{K}(t)$ come together the interior of $\hat{K}(t)$ has a nonempty intersection with G so that the transversality condition can only be satisfied once and therefore there is only one time, namely t^*, for which an extremal controller can steer to G and satisfy the transversality condition. Thus any such extremal controller satisfying the transversality condition is an optimum controller.

Q.E.D.

We have therefore reduced the problem of finding an optimum controller for the approximation problem to that of finding a solution to the two point boundary value problem as given by the 2n+2 equations:

$$\dot{x}^{\circ} = F(x)$$

$$\dot{x} = A(t)x + B(t) \max_{u \in \Omega} \{\eta(t)B(t)u\}$$

$$\dot{\eta} = -\eta A(t) - \eta_o \frac{\partial F'}{\partial x}(x)$$

$$\dot{\eta}_o = 0 \qquad (\eta_o \leq 0)$$

with boundary conditions $\hat{x}(t_o) = \hat{x}_o$, $\hat{x}(t^*) \epsilon\, \partial G$ with $\hat{\eta}(t^*)$ an interior normal to G at $\hat{x}(t^*)$.

3) <u>An Example of Approximate Bounded Phase Coordinate Time Optimal Control</u>

We shall consider a very simple example to illustrate some of the theory of the previous section. Consider a simple mechanism with position coordinate x and velocity coordinate y. Suppose it is desired to bring the mechanism to rest by means of a thrust force u(t) whose magnitude is bidirectional but limited to be less than 1 in magnitude and suppose the velocity is not to exceed .6 in magnitude. That is, consider the linear system

$$\dot{x} = y$$
$$\dot{y} = u(t)$$

with $|u(t)| \leq 1$, $\Lambda = \{x,y \mid |y| \leq .6\}$, $x(0) = 10$, and $y(0) = 0$.

$$\text{Pick } F(x,y) = \tfrac{1}{2}(y - \tfrac{1}{2})^2 \qquad \text{for } y \geq \tfrac{1}{2}$$

$$= 0 \qquad \text{for } |y| \leq \tfrac{1}{2}$$

$$= +\tfrac{1}{2}(y + \tfrac{1}{2})^2 \qquad \text{for } y \leq -\tfrac{1}{2}$$

We shall later determine the parameter $\beta > 0$ so that the strict bound on y is not exceeded. Problems in which the bound is soft are more easily handled since then we can generally pick β ahead of time and in a straightforward manner solve the two point boundary value problem. Here we have picked $F(x,y)$ so that we are constraining the response even before the boundary of Λ is exceeded in hopes of maintaining the strict bound on y. To solve this approximate problem it is merely required that we find a solution of the system:

$$\dot{x}^{\circ} = F(x,y)$$

$$\dot{x} = y$$

$$\dot{y} = \max_{u \in \Omega} \{\eta_2 u\}$$

$$\dot{\eta}_o = 0 \quad (\eta_o \leq 0)$$

$$\dot{\eta}_1 = 0$$

$$\dot{\eta}_2 = -\eta_1 - \eta_o \frac{\partial F}{\partial y}$$

with $x^{\circ}(0) = 0$, $x(0) = 10$, $y(0) = 0$, $x^{\circ}(t_1) \leq \beta$, $x(t_1) = 0$, $y(t_1) = 0$ for some $t_1 > 0$.

A simple calculation shows that picking $\beta = .08$, $\eta_o(0) = -10$, $\eta_1(0) = -1$, $\eta_2(0) \approx -.55$ provides a time optimal solution for this problem. A plot of this response is given by figure 1. Note in this problem the exact optimum solution was obtained, but in general one would pick different $F(x,y)$'s to get better approximations.

4) Remarks on the approximate bounded phase coordinate problems with integral cost

As before consider the linear control process

$\mathcal{L}$) $\dot{x} = A(t)x + B(t)u(t)$

satisfying the conditions stated at the beginning of section 1. As a cost functional of control consider

$$C(u) = g(x(T)) + \int_{t_o}^{T} \{f^o(x,t) + h^o(u,t)\}dt$$

where T = fixed time $> t_o$ and the real functions $f^o(x,t)$ and $h^o(u,t)$ are continuously differentiable and $f^o(x,t)$ is a convex function of x for each t.

The problem of optimal control is to pick an admissible controller $u(t)$ on $[t_o,T]$ so that the response $x_u(t)$ of $\mathcal{L}$ moves from x_o to a target set $\tilde{G} \subset R^n$ at T, ($\tilde{G}$ may be whole space) and minimizes $C(u)$ with the entire response $x_u(t)$ contained is the closed convex restraint set Λ.

As before we introduce the convex differentiable function $F(x)$ satisfying the conditions

$$F(x) > 0 \quad \text{if } x \notin \Lambda$$
$$= 0 \quad \text{if } x \in \Lambda$$

The approximation problem is obtained by adding $F(x)$ to the integrand of the cost functional $C(u)$ to obtain a new cost functional

$$C_\lambda(u) = g(x(T)) + \int_{t_o}^{T} \{f^o(x,t) + \lambda F(x) + h^o(u,t)\}dt$$
$$= \int_{t_o}^{T} \{\tilde{f}^o(x,t) + h^o(u,t)\}dt,$$

here $\lambda \geq 0$. If λ is sufficiently large then one would expect that the contribution from the term $\lambda F(x)$ can be small only if the response stays near Λ or within it. The approximation problem is to find that controller $u(t)$ which minimizes $C_\lambda(u)$ and steers to $\tilde{G} \subset R^n$.

We shall assume that $h^\circ(u,t)$ is convex in u for each t or that the controller is bounded and h is a positive function of u for each t. In either case the previous theory can be applied after slight modification by noting that $\tilde{f}^\circ(x,t) =$ $= f^\circ(x,t) + \lambda F(x)$ is a convex function of x for each t since both f° and F were convex functions and by noting the contribution to $x^\circ(T)$ made by the terms $h^\circ(u,t)$. That is, the problem has now been cast as one which is covered by the sufficiency results of reference 5 which are also necessary [reference 7] and can be obtained as a slight modification of the results of section 2.

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